Econ 8106 MACROECONOMIC THEORY

Prof. L. Jones

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Class Notes: Parts III and IV

These notes deal with two 'tweaks' on the standard single sector growth model:

1. Endogenizing the rate of growth in the system (this is the content of Part 3 of the notes),
2. Introducing the effects of Uncertainty into the models (this is the content of Part 4 of the notes).

Number 1 above deals with simple versions of a literature that began emerging in the 1980's with Paul Romer's dissertation. It deals with a simple problem with the Solow/Cass/Koopmans model of 'growth' – namely that the system does NOT grow unless it is assumed that the production function itself grows. That approach (i.e., simply assuming that the production function changes over time) is useful for some things, there are also difficulties with it:

1. Does this exogenous change in the production function take place everywhere in the world?

2. If yes, does it occur at the same rate everywhere?

3. If yes, why is it that some countries are so poor?

4. If yes, why is measured ’productivity’ in many parts of the world so low?

5. If yes, one implication is that EVEN HOLDING INPUTS FIXED, output should grow. Does this mean that countries in which output grows at less than 2% per year for long periods are employing fewer and fewer resources over time? (This is what would seem to be required.)

6. If this exogenous rate of movement in the production frontier does not occur at the same rate everywhere, why not? And why is it lower in some places than in others?

7. Why is it that the change in the production frontier uses no ’resources’? I.e., if this is supposed to capture the effects of ongoing R&D and/or improvements in education, etc., don’t those activities take up labor and capital to perform?

8. If the answer to '7' is yes, they do take up resources, why aren't they included in the model? What are the incentives to use them up, etc.?

Models of Endogenous Growth are supposed to be a first attempt at a positive solution to those problems. The versions studied below give a simple introduction to the literature, and at the same time, provide a nice review of all of the topics studied under the Exogenous Growth approach, viz., What are the models? What are their comparative statics properties? What sort of time series do they generate and how are those affected by the parameters of preferences and technology? How do you use DP to solve them? What are the effects of policies in the models? What is optimal policy? Etc.

Part 4 deals with a second, logically separate, issue with the single sector growth model. The time series generated by the model are very 'smooth.' That is, they do not fluctuate up and down like those seen in the time series of the US economy (for example). A natural way to change the models to capture something like this is to add random shocks to the models. In principle, this could be done to either the Exogenous Growth, or Endogenous Growth versions of the model. Adding shocks to the Exogenous Growth version of the model gives rise to what is known as the 'Real Business Cycle' model, which has been hugely successful as a modeling device. However, a drawback is that almost nothing can be done analytically for that class of models. That is, almost everything must be done through computer simulations. Because of this, it is less useful as a pedalogical device for learning the ins and outs of adding stochastic elements to the models. Because of this, I always like to introduce the topic of stochastic growth by examining the effects of adding uncertainty to the simplest form of Endogenous Growth model. (These basically become random interest rate savings models.) In this case, if the shocks are i.i.d., we can get some analytical results about the effects of changes in uncertainty in savings rates, consumption shares and wealth growth rates that are simply not possible in Exogenous Growth models.

As an aside on this, you should think carefully about the 'type' of uncertainty that is introduced. That is, what is introduced is 'Aggregate Shocks.' These are shocks that everyone in the economy is subject to. You should convince yourself that if the shocks are 'idiosyncratic,' and there are no barriers to insurance markets, we are back in the non-stochastic world (unless there is an aggregate component to the idiosyncratic shocks – and then, it's no longer clear what it means for the shocks to be 'idiosyncratic'). It is NOT clear what real world phenomenon these 'Aggregate Shocks' are supposed to stand in for. That is, there is no simple, readily measureable microeconomic counterpart to them that can be pointed to. This weakness is there for both the Endogenous and Exogenous Growth versions of the models. What is known is that it is impossible, so far, to replicate the types of aggregate volatility seen in real world time series in a realistic way without introducing them. One possibility is that the shocks are to government policy, and since these shocks affect everyone, they are indeed 'Aggregate.' Although this identification has been attempted, I would not say that it has been 'successful' to this point – if the production function is not subject to direct random shocks, but observed volatility is due to changes in policies, the measured 'A' term in the production function should not exhibit volatility.

1 Part 3: Endogenizing the Growth Rate

1.1 Review

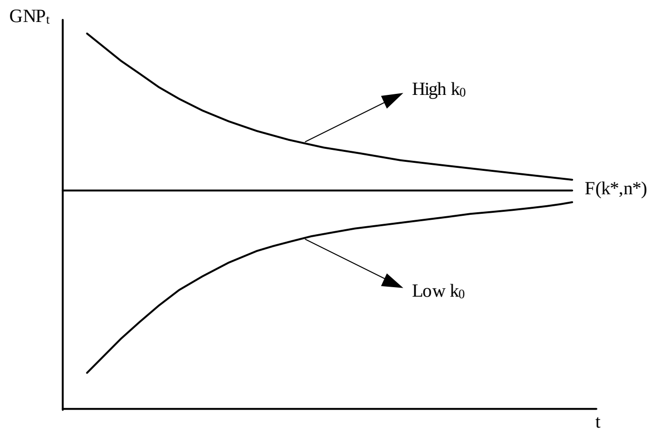
The key feature of the time path of US GNP over the 1950 to 2000 period is that it has exhibited remarkable growth. This is true for many of the countries in the world, and particularly true of the 'developed' countries. Indeed, it is THE thing that distinguishes the developed countries from those that have not developed. For some reason, some countries have had GNP grow, while others have not. Moreover, for the expanding group of 'currently developing' countries, there has been growth, but for some reason, that growth began later in time, and has lagged behind that of the developed world.

What does the model we have studied to this point say about this phenomenon/puzzle? The standard neoclassical growth model is:

$$
\begin{aligned}
&\underset{c,n,x,k}{Max} \quad \sum \beta^t U\left(c_t, 1-n_t\right) \\
&\text{s.t. } c_t + x_t = F\left(k_t, n_t\right) \\
&\qquad k_{t+1} \leq \left(1-\delta\right) k_t + x_t
\end{aligned}
$$

The time paths generated by the solution to this model are shown below.

This version of the model has difficulty when faced with real data. It shows levels of GNP converging to a constant, steady state level, independent of initial conditions. Thus, there is only growth in transition to the steady state, and only for those countries for which the initial capital stock is below the steady state level. This is in contrast to the time series of GNP per capita

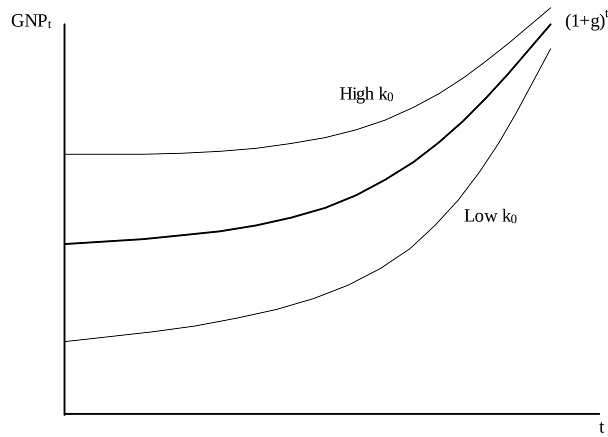


in the developed world. Moreover, in the real world, there seems to be no sign of growth slowing down.

The standard 'fix' to the neo-classical model for this shortcoming is to add exogenous technological change. Sometimes this is assumed to be simply labor augmenting (i.e., multiplying labor supply), sometimes it is assumed to be Harrod neutral (i.e., multiplying the entire production function). It is always taken to be exogenous to the efforts, decisions of the agents in the model. Moreover, it is always assumed to be FREE, that is, it does NOT require any resources. The typical form for this is:

(Exogenous labor supply growth)

$$\underset{c,n,x,k}{Max} \qquad \sum \beta^t U\left(c_t, 1-n_t\right)$$



s.t. $c_t + x_t = F\left(k_t, (1+g)^t n_t\right)$

$$k_{t+1} \leq (1-\delta)\, k_t + x_t$$

where F is a time stationary production function. (Note that if F is Cobb-Douglas, this is equivalent to the Harrod neutral form of technological change.)

What types of time paths are generated for this version of the model? As you have probably already seen, so long as preferences are of the CES form, this model can be 'detrended' by dividing all variables through by $(1+g)^t$ (except for n) resulting in a model that is equivalent to the time independent one as above. (The discount factor and price of new investment goods must be adjusted as well.)

The above model gives better result than the previous one. There is trend growth in the time series that it generated, but this occurs only because it is assumed to grow. Indeed, note that even if, under the counter-factual that k, did not grow, output would still grow in this world. That is, it is impossible for output to not grow!!! You would have to have either k or n shrink over time (or both). This is difficult to understand in a world in which some countries have still not started to grow (e.g., many in Sub-Saharan Africa), many did not start to grow until the 1950's or 1960's, etc. Does this mean that those countries actually had shrinking capital stocks during those periods? Again, this seems implausible at best.

It is also difficult to understand things like the productivity slowdown which took place in the US and other developed countries beginning sometime between 1969.and 1974 (1974 is the usual date given) and 1990 (or so), as well as 'crossings' in levels of GNP per capita by different countries.

For example, If we observe two different countries and we assume that g is common to them both, but that they have different k_0, then both converge to $(1+g)^t F(k^*, n^*)$ This does seem to occur in some examples, e.g., US & Japan. But, in others, the opposite seems to happen. A good example is Japan (or Korea) vs. Argentina. In 1950, GNP per capita was higher in

Argentina than in Japan, or Korea, but that is no longer true. Why does this kind of thing occur? In particular, it's hard to rationalize with the standard model.

This view also creates problems with measured interest rate observations.

An alternative is to assume that the $(1 + g)^t$ term in the production function depends on the country. This also seems problematic. If it depends on the country, it what sense is it 'exogenous'? It's 'exogenous' but also exogenously different in different countries? This gives a simple answer to the question of why some countries produce a lot and others do not. Those that do not, are not capable. But it seems a rather 'hollow' explanation.

Because of this, a new literature has sprung up, beginning with Paul Romer's dissertation (published in 1986). This literature attempts to make the growth rate itself, or interpreted broadly the rate at which the 'technology' is advanced, an endogenous property of the model. All of the models in this class feature a technology set that is independent of time (and country too typically), you can think of this as saying what it is possible to do, with the choices within it being different in different times and countries. The simplest way to think about it is that what you can do in period t in country i as being dependent on the level of 'knowledge' in period t and country i. They are all explicit about how this 'knowledge' evolves over time, and the fact that it requires resources to 'move it' from period t to period t+1. The different models differ in the form that knowledge takes and in how it is transmitted across individuals, times and locations. There is a fundamental question here: Is economically productive knowlegde private thing or public. Or is it a combination of both?

This approach thus offers a very different answer to the question of rich and poor countries then. In rich countries, there is a high level of knowlegde, while in poor, it is low. This also changes the nature of the discussion about development. The key questions become: What causes knowledge to change over time? How is this 'growth' affected by differing incentives? Why is it different in different places? Why does it seem to be linked to 'country' borders rather than general geography or race?

This is now a large, and I think it's fair to say, to this point, empirically unsuccessful literature. Since our time is limited, we will talk only about the simplest of all of the models in this class.

A second difficulty with the model as it stands is that the time paths that it delivers are much 'smoother' than those seen in the data. That is, there are no 'business cycle' fluctuations coming out of the model– i.e., recessions and booms. Three approaches have been forwarded to address this shortcoming:

A) "Endogenous cycles" approach - i.e. chaotic dynamics and dynamical systems and generate complex dynamics from deterministic systems. I think it's fair to say that this literature has not been very empirically successful to this point, although its proponents might disagree, and certainly people are still actively pursuing this line. For example, to generate the kind of behavior seen in US time series, this typically requires extremely low discount factors, making a period more like a generation rather than a quarter. You can play with this yourself some. Imagine what the time series from a growth model would look like if the policy function, $k' = g_k(k)$ was decreasing near the steady state. Draw yourself some pictures and see what you can get out of it as a time series!

B) Aggregate shocks to technology - take the standard growth model and hit it with a series of stochastic shocks:

$$\underset{c,n,x,k}{Max} \qquad E_0\left[\sum \beta^t U\left(c_t, 1-n_t\right)\right]$$

s.t. $c_t + x_t = F\left(k_t, n_t; s_t\right) \left(= s_t k_t^{\alpha} n_t^{1-\alpha}\right)$

$$k_{t+1} \leq (1-\delta)\, k_t + x_t$$

Where s_t is a stochastic process for productivity. (You could also add the growth component to s_t.)

This is what is known as the Real Business Cycle Model I think it is fair to say that this approach has been a qualified success in understanding the high frequency fluctuations seen in US time series. That is, if you measure s_t by assuming that F is Cobb–Douglas and that n_t and k_t are perfectly observed (and α is known), the model has implications for the time series properties of GNP, etc. that are similar to what is seen in the data. (This measurement of the s_t process is called the Solow Residuals.) For example, an implication is that investment should be much more volatile over the cycle than consumption, etc. (Because U is concave, people don't want c to fluctuate much, and hence, systematically plan their investment timing to insure this.)

This is not to say that the model is perfect, or fully satisfactory, but it has some real successes!

Difficulties include what the shocks 'are,' why they move the way they do in the data, why are they common to everyone in the economy, etc. (For example, you can check that if the s_{it} are independent across i then this model does not generate any fluctuations in aggregates whatsoever.)

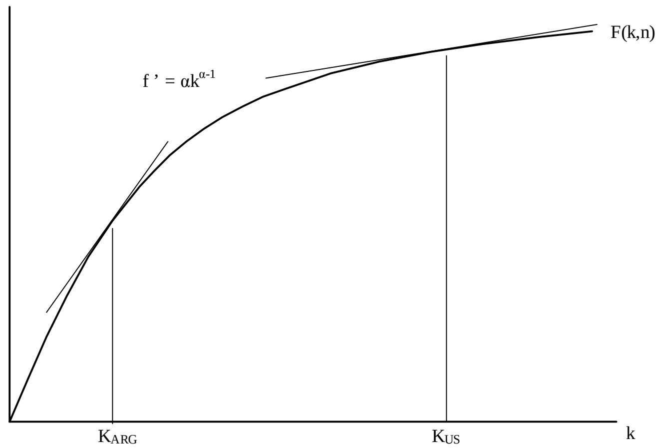
C) Partially in response to the difficulties described in B), many researchers have attempted to try and use the model in B), but with only measured aggregate shocks. Examples of these kinds of shocks are policy shocks such as random fiscal policy and monetary policy.

For example, if taxes were random, then s_t replaced by $1-\tau_t$ in the model. Similarly, random monetary policy could be introduced into the model with the fluctuations given by those actually seen in the data. This literature has also had some moderate successes, but I think it's fair to say that no one has yet found measured, micro-founded shocks that could be used in place of the measured Solow Residuals as outlined in B). The search goes on to either replace them, or link them to some real, observed innovations.

1.2 The Exogenous Growth 'Fix': Labor Augmenting Technological Change

Suppose that the the feasibility constraint is given by:

$$c_t + x_t = F\left(k_t, (1+g)^t n_t\right) \quad \text{(Labor Augmenting technological change)}$$



<Weakness>

i) Can't get differences in long-term growth rates (within a country/between countries)

ii) Even with differences in policy (Mid-term Q.3. change in tax rate τ has no effect on growth rate, $\frac{\partial \gamma}{\partial \tau} = 0$)

iii) Implications about interest rates and k's in US and Argentina

$$\frac{Y_{US}}{Y_{ARG}} = \frac{AK_{US}^{\alpha} N_{US}^{1-\alpha}(1+g)}{AK_{ARG}^{\alpha} N_{ARG}^{1-\alpha}(1+g)} = \left(\frac{K_{US}}{K_{ARG}}\right)^{\alpha} \quad (N_{US} = N_{ARG} = 1)$$

If difference in GNP was roughly 5 times, then

$$5 \simeq \frac{Y_{US}}{Y_{ARG}} = \left(\frac{K_{US}}{K_{ARG}}\right)^{\alpha}$$

Let $\alpha = \frac{1}{3}$, then $\left(\frac{K_{US}}{K_{ARG}}\right) = 5^{\frac{1}{\alpha}} = 125$

$$\frac{f_{US}^{'}}{f_{ARG}^{'}} = \left(\frac{K_{US}}{K_{ARG}}\right)^{1-\alpha} = \left(\frac{K_{US}}{K_{ARG}}\right)^{\frac{2}{3}} = \left(\frac{1}{125}\right)^{\frac{2}{3}} = \frac{1}{25}$$

$$f_{ARG}^{'} = 25 \times f_{US}^{'}$$

(implication, if $r_{US} = 0.1$ then $r_{ARG} = 2.5$. ???)

2 Endogenizing the $(1+g)^t$

Recall from above that the exogenous growth model has a feasibility constraint given by:

$$c_t + x_t \leq y_t = F\left(k_t, (1+g)^t n_t\right)$$

where $(1+g)^t$ is taken as exogenous. Here then, total labor services, or, '*effective hours*', are given by $(1+g)^t n_t$ where n_t is the number of hours spent working. It is important to note that a second thing is true about this formulation, not only is the growth rate of productivity of labor exogenous, it is also *free*! That is, it does not take anything away from the feasibility constraint to have labor productivity grow.

In this part of the notes, we endogenize the growth rate of productivity, or effective hours by introducing human capital to the model. Here, the $(1+g)^t$ term is chosen by the agent, and it will go by the term h_t. Moreover, it is costly to increase h_t – it enters the left hand side of feasibility.

The simplest and most straightforward way to handle this is to have the constraint set given by:

$$c_t + x_{kt} + x_{ht} \leq y_t = F\left(k_t, h_t n_t\right);$$

$$h_{t+1} \leq (1 - \delta_h)h_t + x_{ht};$$

$$k_{t+1} \leq (1 - \delta_k)k_t + x_{kt};$$

(h_0, k_0) fixed.

The model that results is known as the $A(k, h)$ model because the model is 'linear' (or homogeneous of degree one) in (k, h).

As you can see from this, the question will now be – under what conditions is it true that the optimal choice of h_t is $(1 + g)^t h_0$? And if this holds, what is it that determines g? These are the questions that we will answer here.

3 <The $A(k, h)$ Model>

A slight variation on the model makes it quite a bit richer, but not too much more difficult. I think of this as the 'Uzawa' model, but it is not exactly the model outlined in Uzawa's original paper. In that paper, he, for some reason, has very different technologies for the accumulation of physical capital and human capital. This version is both simpler, and more 'standard.'

(1) $\quad Max \quad \sum \beta^t U\left(c_t, 1-n_t\right)$

s.t. $c_t + x_{kt} + x_{ht} \leq F\left(k_t, z_t\right)$

$z_t \leq n_t h_t$ (effective labor)

$k_{t+1} \leq \left(1-\delta_k\right) k_t + x_{kt}$ (physical capital)

$h_{t+1} \leq \left(1-\delta_h\right) h_t + x_{ht}$ (human capital, knowledge)

(1) $F\left(k_t, z_t\right) = A k_t^\alpha z_t^{1-\alpha}$

Then $y_t = A k_t^\alpha \left(n_t h_t\right)^{1-\alpha}$

$$= A k_t^\alpha \left[n_t \{(h_t)^{\frac{1}{t}}\}^t\right]^{1-\alpha}$$

i.e., $h_t = (1+g)^t$ is one interpretation.

(2) OR the alternative:

$Max \quad \sum \beta^t U\left(c_t, (1-n_t) h_t\right)$

s.t. $c_t + x_{kt} + x_{ht} \leq F\left(k_t, z_t\right)$

$z_t \leq n_t h_t$

$k_{t+1} \leq \left(1-\delta_k\right) k_t + x_{kt}$

$h_{t+1} \leq \left(1-\delta_h\right) h_t + x_{ht}$

This version assumes that h_t also affects utility. It's hard to know exactly what this means, but one possibility is that the 'leisure' term in utility really captures home production, and that a higher level of h_t also makes one more productive in the home, not just in market activities. This formulation comes originally from Heckman (1976) and has better mathematical properties – it is easier to give 'decentralizations' of the solution to the planner's problem in this case. If labor is inelastically supplied, the two versions are equivalent.

NOTE: . Now, $1+g$ is determined by h with the equation $c_t+x_{kt}+x_{ht} \leq F(k_t, z_t)$ and $h_{t+1} \leq (1-\delta_h) h_t + x_{ht}$.

(3) Just for reference purposes, the original Uzawa model was:

$$Max \qquad \sum \beta^t U(c_t, 1 - n_{m_t} - n_{h_t})$$

s.t. $c_t + x_{k_t} \leq F(k_t, n_{m_t} h_t)$

$$k_{t+1} \leq (1-\delta_k) k_t + x_{kt}$$

$$h_{t+1} \leq \psi(n_{h_t}) h_t$$

(4) Is h_t individual or social? What we have done here takes h_t as individual. In some models, h_t is assumed to depend on others' h_t – there is an externality.

One way to incorporate this would be:

$$h_{it+1} \leq (1-\delta_h) h_{it} + G\left(x_{ih}, \int_I x_{i'h} di'\right)$$

Literally, this would mean that how much you learned in a period – $h_{it+1} - (1-\delta_h) h_{it}$ depends not only on how much effort you personally put into learning – x_{ih}, but also how much effort others put in as well – $\int_I x_{i'h} di'$. It is not clear how much sense this makes, or indeed, why this versus any other formulation might be 'good.' For example, what if G is a function of $\int_I x_{i'h} di'$ only? Then nobody will invest $x_{ih} = 0$. It follows that $h_{it} \to 0$ under this specification. Thus, this must be done carefully.

3.0.1 Special Case of $A(k,h)$ model

We'll do the simplified version with inelastic labor supply, $n_t = 1$

$$(P) \qquad Max \qquad \sum \beta^t U(c_t)$$

s.t. $c_t + x_{k_t} + x_{h_t} \leq AF(k_t, h_t)$

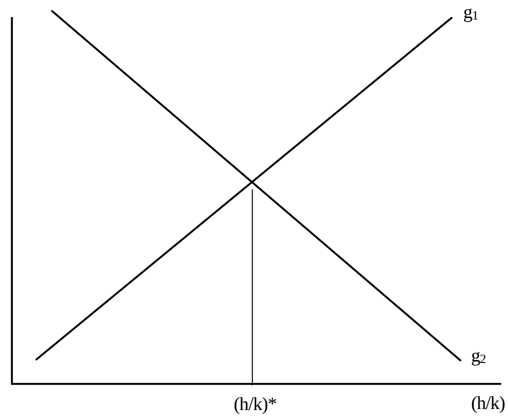
$$k_{t+1} \leq (1-\delta_k) k_t + x_{kt}$$

$$h_{t+1} \leq (1-\delta_h) h_t + x_{ht}$$

(FOC)

$$\text{(EEK)} \qquad U'(c_t) = \beta U'(c_{t+1}) (1 - \delta_k + F_k(t+1))$$

$$\text{(EEH)} \qquad U'(c_t) = \beta U'(c_{t+1}) (1 - \delta_h + F_h(t+1))$$



(Note $\frac{\partial z_{t+1}}{\partial h_{t+1}} = n_{t+1} = 1.$)

NOTE: If labor enters the utility function, then $\frac{\partial z}{\partial n}$ remains in the equation and in EEH, $F_z(t+1)\,n_{t+1}$.

Thus, $1 - \delta_k + F_k(t+1) = 1 - \delta_h + F_h(t+1)$. This is of the form $g_1\left(\frac{h}{k}\right) = g_2\left(\frac{h}{k}\right)$ since F is homogeneous of degree one and doesn't depend on time. Graphically,

Remark:

Notice that $F_k(t+1) = F_k(k_{t+1}, h_{t+1}) = F_k(\frac{k_{t+1}}{h_{t+1}}, 1)$ since F_k is homogeneous of degree zero. Since F is concave, it follows that $F_k(\frac{k}{h}, 1)$ is decreasing in $\frac{k}{h}$, hence, g_1 is increasing in h/k.

Similarly, $F_h(t+1) = F_h(k_{t+1}, h_{t+1}) = F_h(1, \frac{h_{t+1}}{k_{t+1}})$ since F_h is homogeneous of degree zero. Again, since F is concave, it follows that $F_h(1, \frac{h}{k})$ is decreasing in $\frac{h}{k}$, hence, g_2 is decreasing in h/k.

This is why the graph looks as it does.

Thus, it follows that there is a unique ratio of h to k, $\frac{h}{k}^*$ and that $\frac{h_t}{k_t} = \frac{h}{k}^*$ for all t. Let's call that ration ψ, so that $h_t = \psi k_t$ for all t.

From this we see that $F(k_t, h_t) = F(k_t, \psi k_t) = k_t F(1, \psi) = A k_t$ where $A = F(1, \psi)$.

As an example, in the Cobb-Douglas case with $\delta_h = \delta_k$, we can give an explicit form for the optimal ratio of h to k.

In this case, $r_t k_t = \alpha y_t$,

so, $F_k(t+1) = \alpha \frac{y_{t+1}}{k_{t+1}}$ $\qquad (\star)$

Also, $w_t h_t = (1-\alpha)\, y_t$

so, $F_z(t+1) = (1-\alpha)\, \frac{y_{t+1}}{h_{t+1}}$ $\qquad (\star\star)$

Since $(\star) = (\star\star)$ at $\left(\frac{h}{k}\right)^*$,

$\alpha \frac{y_{t+1}}{k_{t+1}} = (1-\alpha)\, \frac{y_{t+1}}{h_{t+1}}$

or $\boxed{\frac{h_{t+1}}{k_{t+1}} = \frac{1-\alpha}{\alpha}}$ $\qquad$ (doesn't depend on t)

From the laws of motion of the two capital stocks, we get:

$$h_{t+1} = (1-\delta_h) h_t + x_{ht}$$

$$\psi k_{t+1} = (1-\delta_h)\psi k_t + x_{ht}$$

$$\psi((1-\delta_k)k_t + x_{kt}) = (1-\delta_h)\psi k_t + x_{ht}$$

$$\psi(\delta_h - \delta_k)k_t = x_{ht} - \psi x_{kt}$$

Now, assume that $\delta_k = \delta_h$ to see that:

$x_{ht} = \psi x_{kt}$ as well.

(In the Cobb-Douglas case, this becomes: $x_{ht} = \frac{1-\alpha}{\alpha} x_{kt}$.)

Problem: What happens if $\delta_k \neq \delta_h$?

These simplifications almost turn the problem into a one capital problem. The reason this is not quite true is that the initial ratio of h to k may not be 'right' and hence we need to carry both stocks along as states even though we know that beginning in period $t = 1$, there will effectively be only one state.

To avoid that extra complication assume also that $h_0 = \psi k_0$.

Thus, the solution to (P) solves:

$$Max \qquad \sum \beta^t U(c_t)$$

s.t. $c_t + (1+\psi)x_{kt} \leq Ak_t$

That is, the $A(k,h)$ model with n fixed is equivalent to what is knows as an Ak model, this is discussed in detail below.

This is a standard one sector growth model at this point except for one fact. This is that the standard Inada conditions on the production function do not hold, i.e., $f(0) \neq \infty$ and $f(\infty) \neq 0$. Because of this difference, what we saw in the standard case about dynamics and stability need not hold.

What does hold? To figure this out, let's make one more assumption – $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$. Thus, the problem is:

$$Max \qquad \sum \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t. $c_t + (1+\psi)x_{kt} \leq Ak_t$,

$k_{t+1} \leq (1-\delta_k)k_t + x_{kt}$,

k_0 given.

This is a homogeneous/homothetic problem. That is, if a path is feasible from the initial condition k_0 then, λ times that path is feasible from the initial condition λk_0. Moreover, multiplying the entire time path of c_t by λ multiplies utility by $\lambda^{1-\sigma}$.

As we have seen before in problems like this, it follows that the optimal path from λk_0 is simply λ times the optimal path from k_0.

Because of this, we have the property of the value function:

$$v(\lambda k) = \lambda^{1-\sigma} v(k)$$

Using this with $\lambda = k$ and $k = 1$ gives us:

$$v(k) = k^{1-\sigma} v(1).$$

Thus, Bellman's equation becomes:

$$v(k) = \sup_{c,x_k,k'} \frac{c^{1-\sigma}}{1-\sigma} + \beta k^{1-\sigma} v(1)$$

s.t. $\quad c + (1+\psi)x_k \leq Ak$

$$k' \leq (1-\delta_k)k + x_k$$

Notice that the problem on the RHS of the BE is also a homogeneous/homothetic problem. The constraint set is homogeneous of degree one in k, while the utility function is homogeneous of degree $1-\sigma$ in (c, k').

Hence, the solution is homogeneous of degree one in the state. E.g., if we denote the policy functions by $c = g_c(k)$, etc., then,

$$g_c(\lambda k) = \lambda g_c(k).$$

As above then, we have:

$$g_c(k) = k g_c(1).$$

I.e., c is linear in k. Similar arguments hold for g_x and g_k.

Note that since $y = Ak$ is also linear in k, it follows that c, and x_k are time invariant fractions of output.

Thus, to completely characterize the solution to the problem, all we need to know are the three slopes, $g_c(1)$, etc.

There are many ways to do this. Let's go back to the original problem:

$$Max \qquad \sum \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t. $c_t + (1+\psi)\left(k_{t+1} - (1-\delta_k)k_t\right) \leq Ak_t,$

k_0 given.

$$Max \qquad \sum \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t. $c_t + (1+\psi)k_{t+1} \leq \left(A + (1+\psi)(1-\delta_k)\right) k_t,$

k_0 given.

$$Max \qquad \sum \beta^t \frac{\left((A+(1+\psi)(1-\delta_k))k_t - (1+\psi)k_{t+1}\right)^{1-\sigma}}{1-\sigma}$$

The FOC for this problem is:

$$\beta^t c_t^{-\sigma}(1+\psi) = \beta^{t+1} c_{t+1}^{-\sigma}\left(A + (1+\psi)(1-\delta_k)\right)$$

$$\left(\frac{c_{t+1}}{c_t}\right)^\sigma = \beta\left(\frac{A}{1+\psi} + (1-\delta_k)\right)$$

$$\left(\frac{g_c(1)k_{t+1}}{g_c(1)k_t}\right)^\sigma = \beta\left(\frac{A}{1+\psi} + (1-\delta_k)\right)$$

$$\left(\frac{g_k(1)k_t}{k_t}\right)^\sigma = \beta\left(\frac{A}{1+\psi} + (1-\delta_k)\right)$$

$$g_k(1) = \left[\beta\left(\frac{A}{1+\psi} + (1-\delta_k)\right)\right]^{1/\sigma}.$$

Using this and the law of motion of k will give you $g_x(1)$, and finally, $g_c(1)$ can be obtained using feasibility:

$$g_k(1)k_t = (1-\delta_k)k_t + g_x(1)k_t$$

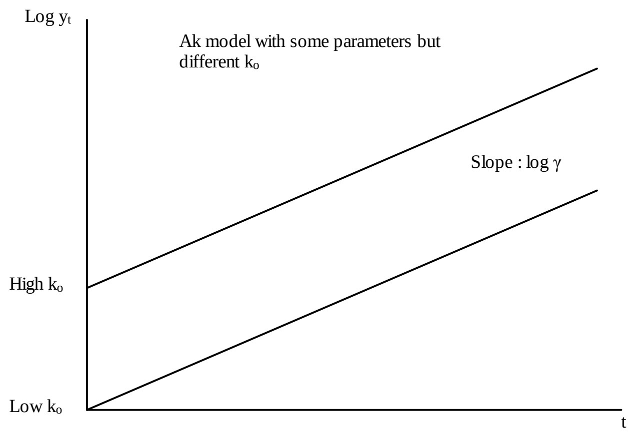
$$g_k(1) = (1-\delta_k) + g_x(1)$$

$$g_x(1) = g_k(1) - (1-\delta_k)$$

$$g_c(1)k_t + (1+\psi)g_x(1)k_t = Ak_t$$

$$g_c(1) = A - (1+\psi)g_x(1).$$

Notice that this model will exhibit growth (endogenously) if and only if $g_k(1) > 1$, i.e.,

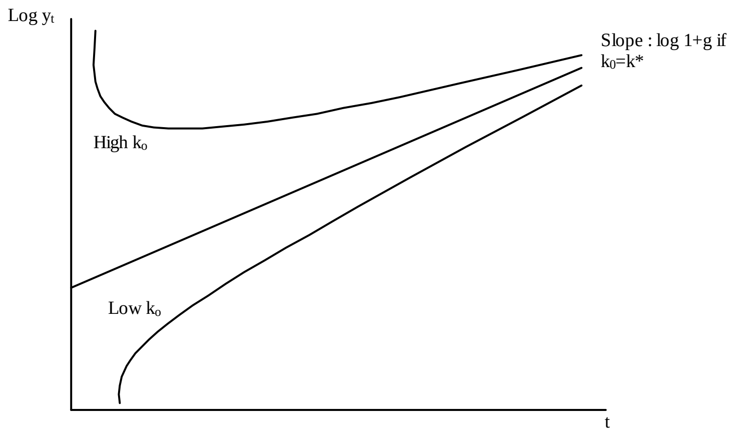


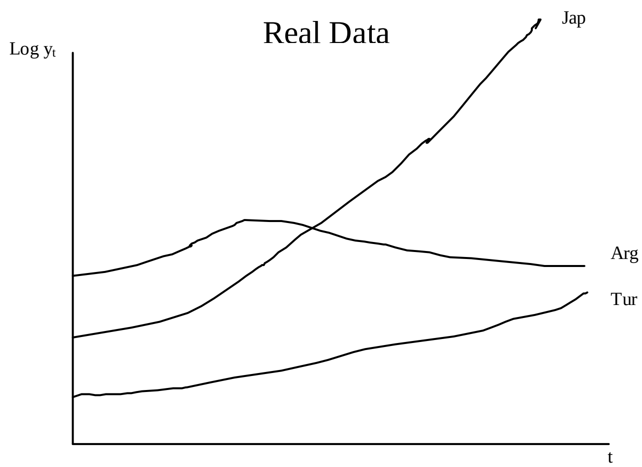
$$\left[\beta\left(\frac{A}{1+\psi}+(1-\delta_k)\right)\right]^{1/\sigma}.$$

This condition is a combination of preference and technology parameters. More productive technologies and more patient households give rise to higher rates of growth. In general, higher σ will imply lower growth rates.

Two countries with different initial capital stock (k_0) but same everything else.

In exogenous growth model,


Log y_t
Slope : log 1+g if $k_0=k^*$
High k_o
Low k_o
t


Real Data
Log y_t
Jap
Arg
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t

3.1 < ***The*** Ak ***Model*** >

For historical reasons, it's useful to know a variant of the model presented above, the Ak model. You can check for yourself that this is exactly what you get above when you assume that $\psi = 0$.

Planner's problem $P(k_0)-$ problem with initial capital given as k_0 :

$$Max \qquad U\left(\underset{\sim}{c}\right) = \sum \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t. $c_t + x_t \leq Ak_t$

$$k_{t+1} \leq (1-\delta)\, k_t + x_t$$

k_0 fixed.

Note that this corresponds to the exogenous growth model with $\alpha = 1$.

Think of this as k representing the individual's knowledge in any given period. Note that under this interpretation, we have adopted the extreme (but simple, and maybe not so bad) assumption that knowledge is a purely private good.

3.2 Solution to the Ak Model

Let $\Gamma(k_0) = \{(c_t, x_t, k_t)_{t=0}^{\infty} \mid feasible\ from\ k_0\}$ be the set of feasible time paths from initial condition k_0.

Then $z \in \Gamma(k_0) \;\Leftrightarrow\; \lambda z \in \Gamma(\lambda k_0) \;\; \forall \;\; \lambda \geq 0$

(Homogeneous of degree 1 in k_0)

(ex. if initial capital doubles, then so does consumption, investment, and capital stock.)

Let's also make the same restriction on U as is typically made in the exogenous growth model, i.e. CES.

$U(\lambda \underset{\sim}{c}) = \lambda^{1-\sigma} U(\underset{\sim}{c})$ ($\rightarrow U$ is homogeneous of degree $1-\sigma$) also homothetic

Thus,

Proposition I) $(c_t^*, x_t^*, k_t^*)_{t=0}^{\infty}$ solves $P(k_0)$

$$\Leftrightarrow (\lambda c_t^*, \lambda x_t^*, \lambda k_t^*)_{t=0}^{\infty} \text{ solves } P(\lambda k_0)$$

Proof: Same as what was done previously when discussing the properties of homothetic utility functions and aggregation.

Proposition II) $V(k)$ is homogeneous of degree $1-\sigma$ in k_0, i.e. $V(\lambda k) = \lambda^{1-\sigma} V(k)$

Proof: Obvious.

Note that $P(k_0)$ is a stationary dynamic programming problem, and hence,

$\Rightarrow \exists$ policy function $g_k(k)$ s.t. if $(k_0^*, k_1^*, ...)$ solves $P(k_0)$ then

i) $k_0^* = k_0 = g_k^0(k)$

ii) $k_1^* = g_k(k_0)$

iii) $k_2^* = g_k(k_1) = g_k(g_k(k_0)) = g_k^2(k_0)$

What does HD 1 tell us about k_1 as a function of k_0?

From Proposition I, it follows that $k_t^*(\lambda k_0) = \lambda k_t^*(k_0)$ for all t. This, in particular,

$k_1^*(\lambda k_0) = \lambda k_1^*(k_0)$

i.e. $g_k(\lambda k_0) = \lambda g_k(k_0)$

so, $g_k(k) = g_k(1 \times k) = k g_k(1)$

i.e. $g_k(k) = \eta_k \cdot k \qquad$ where $\eta_k = g_k(1)$

Stationary dynamic programming implies $\exists$ g_k, g_x, and g_c such that

$(\star)$ $x_0^* = g_x(k_0)$

$x_1^* = g_x(k_1)$

$x_2^* = g_x(k_2)$

$(\star\star)\ c_0^* = g_c(k_0)$

$c_1^* = g_c(k_1)$

$c_2^* = g_c(k_2)$

$(\star) \Longrightarrow k_1^* = (1-\delta)\,k_0 + x_0^*$

$\rightarrow \eta_k k_0 - (1-\delta)\,k_0 = x_0^*$

$\therefore\ \ g_x(k) = (\eta_k - (1-\delta))\,k_0$ holds for all k (let $\eta_x = \eta_k - (1-\delta)$)

$(\star\star) \Longrightarrow c_0^* = Ak_0 - x_0^* = Ak_0 - \eta_x k_0 = (A - \eta_x)\,k_0$ (let $\eta_c = A - \eta_x$)

So, the solution to the Ak model is determined by three 'policy functions'.

$k' = g_k(k) = \eta_k \cdot k$

$c = g_c(k) = \eta_c \cdot k$

$x = g_x(k) = \eta_x \cdot k$

and $\eta_x = \eta_k - (1-\delta)$ & $\eta_c = A - \eta_x$

We need only to figure out what the constants η_k, η_c, and η_x are.

(EE) from Ak model says: (Assuming $\eta_x > 0$)

$\frac{U'(c_t)}{U'(c_{t+1})} = \beta\,(1-\delta+A)$ where $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$

i) LHS doesn't depend on time

ii) $U'(c_t) = c_t^{-\sigma}$

iii) LHS $\rightarrow \left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\,(1-\delta+A)$

$\Longrightarrow$ growth rate of consumption depends on σ, β, δ, A (not exogenously determined).

Since $c_{t+1} = \eta_c \cdot k_{t+1} = \eta_c \cdot \eta_k \cdot k_t$

$\left(\frac{\eta_c \cdot \eta_k \cdot k_t}{\eta_c \cdot k_t}\right)^{\sigma} = \beta\left(1 - \delta + A\right)$

so, $\eta_k = \left[\beta\left(1 - \delta + A\right)\right]^{\frac{1}{\sigma}}$

Then, $\gamma_c = \frac{c_{t+1}}{c_t} = \frac{\eta_c \cdot \eta_k \cdot k_t}{\eta_c \cdot k_t} = \eta_k$

$$\gamma_k = \frac{k_{t+1}}{k_t} = \frac{\eta_k \cdot k_t}{k_t} = \eta_k$$

$$\gamma_x = \frac{x_{t+1}}{x_t} = \frac{\eta_x \cdot \eta_k \cdot k_t}{\eta_x \cdot k_t} = \eta_k$$

$\boxed{\therefore \gamma_c = \gamma_k = \gamma_x = \gamma}$

Remark:. This last part assumes that the solution is interior. This may seem fine, but there are cases where this is difficult, for example, if this were to be consistent with the Chad case $\beta\left(1 - \delta + A\right) < 1$.

3.3 Comparative Statics of Growth in the Ak Model

- $\frac{\partial \gamma}{\partial A} > 0$

(rmk. $x = \eta_x \cdot k = \frac{\eta_x}{A} Ak$)

- $\frac{\partial \gamma}{\partial \delta} < 0$

- $\frac{\partial\gamma}{\partial\beta} > 0$ (make people more patient and growth rate increases)

- $\frac{\partial\gamma}{\partial\sigma}$? ($\sigma$ =elasticity of consumption between two periods)

$\frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} = \frac{x_{t+1}}{x_t} = \frac{Y_{t+1}}{Y_t} = \gamma = [\beta(1-\delta+A)]^{\frac{1}{\sigma}}$

- $\frac{\partial\gamma}{\partial\sigma} < 0$ (if $\gamma > 1$)

- $\frac{\partial\gamma}{\partial\sigma} > 0$ (if $\gamma < 1$)

(σ is intertemporal rate of substitution, or risk averseness, i.e. if $\sigma \uparrow$ then more risk averse. $\sigma = 0$ then linear indifference curve, $\sigma = 1$ then log i/c, $\sigma = \infty$ then Leontief)

3.3.1 Remarks:.

1. Growth from a *time stationary* technology (endogenous growth models - as opposed to exogenous growth model where $F(k,n;t) = A'k^{\alpha}\left((1+g)^t n\right)^{1-\alpha}$), $F(k) = Ak$.

2. Remember that we want to think of k as standing for knowledge not physical capital.

3. γ depends on 'deep' parameters of technology & preferences in the model $(\sigma, \beta, \delta, A...)$

3.4 Adding Taxes in the Ak Model (τ_{ct}&τ_{kt})

$$Max \qquad \sum \beta^t U(c_t)$$

s.t. $\sum p_t \left[(1+\tau_{ct})c_t + x_t\right] \leq \sum \left[(1-\tau_{kt}) r_t k_t + T_t\right]$

$$k_{t+1} \leq (1-\delta) k_t + x_t$$

(EEK) $\frac{U'(c_t)}{1+\tau_{ct}} = \frac{\beta U'(c_{t+1})}{1+\tau_{ct+1}} \left[1-\delta+(1-\tau_{kt+1}) r_{t+1}\right]$

$$\frac{U'(c_t)}{U'(c_{t+1})} = \frac{\beta(1+\tau_{ct})}{1+\tau_{ct+1}} \left[1-\delta+(1-\tau_{kt+1}) A\right] \quad (\because A = \frac{\partial F}{\partial k_{t+1}})$$

$$\frac{U'(c_t)}{U'(c_{t+1})} = \left(\frac{c_{t+1}}{c_t}\right)^{\sigma} \qquad \text{if } U(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

Thus, if τ_{ct} and τ_{kt} are constant, we have:

$\gamma = \left[\beta\left(1-\delta+(1-\tau_k) A\right)\right]^{\frac{1}{\sigma}}$ (also constant)

Note: Now the growth rate depends on fiscal policy through the tax rate τ_k.

(1) Does it matter if the revenue is used to finance transfers, T, or government spending, g?

if $T_t = \tau_{kt} r_t k_t + \tau_{ct} p_t c_t$ $\quad (FP\ A) \quad \gamma_A$

or if $p_t g_t = \tau_{kt} r_t k_t + \tau_{ct} p_t c_t$ $\quad (FP\ B) \quad \gamma_B$

$$\gamma_A = \gamma_B$$

Since the growth rate is directly determined through the Euler Equation, we can see that both of these policies will have the same growth properties. However, utility is higher if $g_t = 0$ i.e. in the case of $FP\ A$. You should try and show this yourself!

$$\gamma^\sigma = \left(\frac{c_{t+1}}{c_t}\right)^\sigma = \frac{\beta(1+\tau_{ct})}{1+\tau_{ct+1}}\left[1-\delta+\left(1-\tau_{kt+1}\right)A\right]$$

$$\tau_{kt} = \tau_k \qquad \forall\ t$$

$$\tau_{ct} = \tau_c \qquad \forall\ t$$

1. $\frac{\partial\gamma}{\partial\tau_c} = 0$

2. $\frac{\partial\gamma}{\partial\tau_k} < 0$

3. You can grow too fast, by setting $\tau_c > 0$ and $\tau_k < 0$ That is, raise revenues from a consumption tax and use this to subsidize (negative tax) capital. In this case,

$$\gamma = \left[\beta\{1-\delta+\left(1-\tau_k\right)A\}\right]^{\frac{1}{\sigma}} \ > \ \left[\beta\{1-\delta+A\}\right]^{\frac{1}{\sigma}} \ \left(=\gamma' \text{ if } \tau_c = \tau_k = 0\right)$$

(speed up growth)

3.4.1 Remarks:

1. Note that this points out that high γ is NOT the same as high U! For example, the policy given causes U to fall even though γ is higher!

2. Actually, in this model, the undistorted growth rate, $[\beta\{1-\delta+A\}]^{\frac{1}{\sigma}} \equiv \gamma^*$ is optimal.

3. Some people have suggested that some spurts of growth actually observed are inefficiently high. This shows you how this could happen in this model. An example might be growth in the Soviet Union during the Stalinist period. Here you take away leisure and force output into investment. This will lead to a high growth rate in output (but not if it's just tanks!!!).

4. Optimal taxation:

   $\tau_k \to 0$ still holds.

5. What about $A(k,h)$ and taxation?

   (Assume inelastic labor supply)

   $(CP) \qquad Max \qquad \sum \beta^t U(c_t)$

   s.t. $\sum p_t [c_t + x_{kt} + x_{ht}] \leq \sum [(1-\tau_{nt}) w_t n_t h_t + (1-\tau_{kt}) r_t k_t]$

$k_{t+1} \leq (1-\delta)\, k_t + x_{kt}$

$h_{t+1} \leq (1-\delta)\, h_t + x_{ht}$

$(EEK)\ \left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\left[1-\delta_k + (1-\tau_{kt+1})\, F_k\,(t+1)\right]$

$(EEH)\ \left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\left[1-\delta_h + (1-\tau_{nt+1})\, F_z\,(t+1)\right]$

Assume that the production function is Cobb-Douglas and that $\delta_k = \delta_h$.

Then, it follows that $(1-\tau_{kt+1})\, F_k\,(t+1) = (1-\tau_{nt+1})\, F_z\,(t+1)$.

A) $\frac{h_{t+1}}{k_{t+1}} = \frac{1-\tau_{nt+1}}{1-\tau_{kt+1}} \cdot \frac{1-\alpha}{\alpha}$

That is, differential tax rates create a distortion on the composition of capital, the ratio of h_{t+1} to k_{t+1}.

In SS, $\tau_{kt} = \tau_k$ & $\tau_{nt} = \tau_n \ \forall\ t$

If $\tau_k = \tau_n$ then $\frac{h_t}{k_t} = \frac{1-\alpha}{\alpha}$. (Same as no-tax case, with optimal ratio $\frac{1-\alpha}{\alpha}$)

B) Can you increase the growth rate by using taxes to move away from $\frac{h_t}{k_t} = \frac{1-\alpha}{\alpha}$?

C) Optimal Taxes (time path of taxes)

$\tau_{kt} \rightarrow 0$

$\tau_{nt} \rightarrow 0$ (labor income tax goes to zero)

6. Then does government revenue go to zero? Yes. So, how are expenditures financed?

   Gov't raises tax in the beginning and lives off from the interest earned.

7. $Max \quad \sum \beta^t U(c_t, 1-n_t)$ (elastic labor)

   s.t. $\sum p_t [(1+\tau_{ct}) c_t + x_{kt} + x_{ht}] \leq \sum [(1-\tau_{nt}) w_t n_t h_t + (1-\tau_{kt}) r_t k_t]$

   $$k_{t+1} \leq (1-\delta) k_t + x_{kt} \qquad h_{t+1} \leq (1-\delta) h_t + x_{ht}$$

8. The 'Too many taxes' proposition that we had in our earlier study in the neo-classical model, which says:

   Get rid of consumption tax and get the same allocation, doesn't hold in this case.

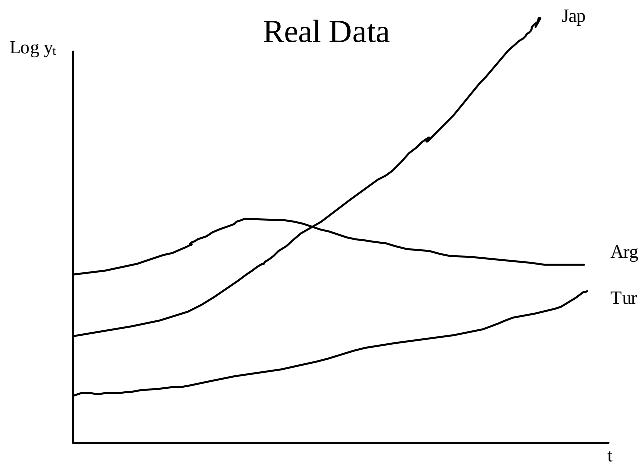
   A) τ_{ct} is no longer redundant.

   B) $\tau_{kt} \to 0$, $\tau_{nt} \to 0$

   C) If $U = c^{1-\sigma} V(l)$ $\tau_{c_t} \to 0$ too.

   D) Here, $\frac{\partial \gamma}{\partial \tau_c} \neq 0$ (rather, usually it is negative.)

9. If τ_c is increased, then typically, you consume less and enjoy leisure more, which leads to a decrease in labor supply. Since the use of human



capital is related to this, this also lowers the rate of return for investing in h. This typically results in a lower growth rate, but in most examples I have seen, this effect is quite small.

With Neoclassical model, it was difficult to get 'crossings' in GDP per capita levels. (Shown is Argentina and Japan.)

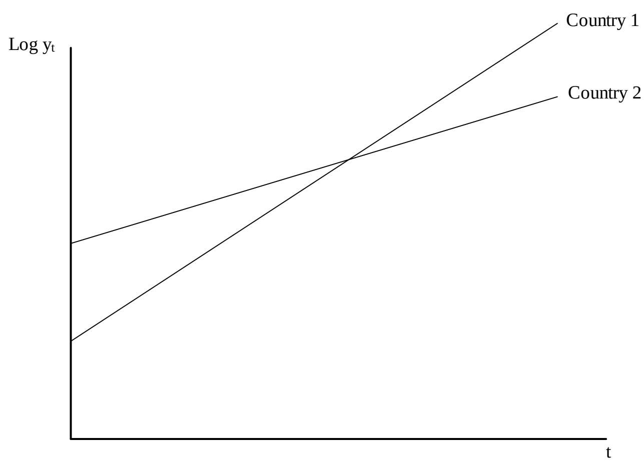
Can Ak model with τ_k generate crossings?

To get the above picture, you need

i) $k_{20} > k_{10}$ (initial capital is higher in country 2)

ii) $\tau_{k2} > \tau_{k1}$ (higher tax rate in country 2)

Note: Unfortunately, typically it is the more advanced countries that have the higher tax rates. This creates difficulty with this as a theoretical



explanation.

Econ 8105 MACROECONOMIC THEORY

Class Notes: Part IV

Prof. L. Jones

4 Part IV: Stochastic Models

4.1 *Adding Wiggles to the Time Series*

As we noted in the previous section, two different approaches have developed in the literature for the fact that the observed time series, unlike those generated by the models studied to this point, are not smooth. The two different ways are deterministic chaotic dynamics, and the explicit inclusion of random elements in the models. The last part of the class notes deals with one specific example of this second approach. In it's more popular form, this approach is the foundation of the modern approach to business cycle frequency fluctuations in output, investment and employment. The prototypical model in this genre is:

4.2 The RBC Model

$Max \qquad E_0 \left[\sum \beta^t U\left(c_t\left(s^t\right), 1-n_t\left(s^t\right)\right)\right]$ (expected discounted value of future utility)

$$\text{s.t.} \qquad c_t\left(s^t\right)+x_t\left(s^t\right)=F\left(k_t\left(s^{t-1}\right), n_t\left(s^t\right); s^t\right)$$

$$k_{t+1}\left(s^t\right) \leq (1-\delta)\, k_t\left(s^{t-1}\right)+x_t\left(s^t\right)$$

Where the $s_0, s_1, ...$ form an infinite sequence of random variables, and $s^t = (s_0, s_1, ...s_t)$ denotes the history up to period t. Note that it is assumed that all choice variables are functions of the entire history of shocks up to and including the date at which the decision is made. It is also assumed that the current date decisions, labor supply, consumption and investment are made in period t after the shock in period t is 'seen.'

This class was first studied theoretically by Brock and Mirman.

The solution to this maximization problem is a stochastic process for the endogenous variables, c, n, x and k and, of course, the solution depends on the properties of the underlying stochastic process for s.

For example, if $s_t \equiv 1$, $\forall t$, then this model is identical to the Neoclassical model without uncertainty. Typically, researchers assume that, $s_t = (1+g)^t \exp(z_t)$ where the z_t are stationary ($AR(1)$ for example).

Pressing further, if we assume that $F\left(k_t\left(s^{t-1}\right), n_t\left(s^t\right); s^t\right) = s_t k_t^{\alpha} n_t^{1-\alpha}$, we can, with data on y_t, k_t and n_t, and knowledge of α, recover the true stochastic process for the s_t. These are known as Solow residuals, $\log(s_t) = \log(y_t) - \alpha \log(k_t) - (1-\alpha)\log(n_t)$. The interpretation here is that the s_t sequence is a sequence of 'technological shocks.' Note that since we have formulated the model as a representative agent problem, it is important that the value of the stochastic shocks are the same for everyone.

The study of this model and its properties has been the object of intense study since the publication of the paper by Kydland and Prescott in 1983. Very few analytic results are available for this model, so most of the work involves considerable simulation.

Since it would take too much time to do the set up, etc., to do justice to this model. I will try and give you an introduction to some of the issues that arise in a simpler version of the model. This is a stochastic version of the Ak model discussed above. Most of the conceptual issues that arise from modelling are similar, and we can actually get out a pretty interesting result analytically that shows how stochastic models differ from their counterparts with no uncertainty.

4.3 The Stochastic Ak model

Consider the maximization problem:

$P(k_0, s_0)$

$Max \qquad E_0\left[\sum \beta^t U\left(c_t\left(s^t\right)\right)\right]$ (expected discounted value of future utility)

s.t. $c_t\left(s^t\right) + x_t\left(s^t\right) \leq A\left(s_t\right) k_t\left(s^{t-1}\right)$

$k_{t+1}\left(s^t\right) = x_t\left(s^t\right)$

Thus, we have assumed that there is full depreciation, $\delta = 1$ (or alternatively that δ is embedded in A), and that the value of A depends only on the current value of the shock, s_t.

Assume further that $U\left(c\right) = \frac{c^{1-\sigma}}{1-\sigma}$.

Let $V(k_0, s_0)$ be the maximized value of $P(k_0, s_0)$, assuming that a solution exists.

Proposition: Under the assumptions above, the solution to $P(k_0, s_0)$ satisfies:

1) $\qquad V(\lambda k_0, s_0) = \lambda^{1-\sigma} V(k_0, s_0)$ Homogeneity of the value function in the intial capital stock.

2) $\left(\underset{\sim}{c}^{*}\left(\lambda k_{0}, s_{0}\right), k^{*}\left(\lambda k_{0}, s_{0}\right)\right)=\lambda\left(\underset{\sim}{c}^{*}\left(k_{0}, s_{0}\right), k^{*}\left(k_{0}, s_{0}\right)\right)$ homogeneity of the optimal time paths (which are state contingent) in the initial capital stock.

Proof: Obvious.

The stochastic process $s_0, s_1, ...,$ is called a First Order Markov Process if $P\left(s_{t+1}=s|s_0=s_0,...,s_t=s_t\right) = P\left(s_{t+1}=s|s_t=s_t\right)$. That is, if the transition probabilies among states depend only on the most recent realization of the state. This is the definition if a process takes on only finitely many (or countably many) values. The definition in the continuous case is the natural analogue where $P's$ are replaced by densities (or in general, by conditional distribution functions).

Examples of stochastic processes satisfying this restriction are i.i.d processes (these are actually zero order Markov processes since their transition probabilities don't depend on ANY of the elements of the history), and AR(1) & MA(1) processes.

Recall that the non-stochastic version of the Neoclassical growth can be simplified into Bellman's equation in the Value of the problem. Here, it is 'as if' you only care about the new k and not about the entire future time path. This is because V already captures the value of the remainder of the path. A similar result holds for stochastic maximization problems as well. For the example we are studying, it is:

Theorem: Assume that $s_0, s_1, ...$ is a First Order Markov Process. Then, (Bellman's equation) $V(k,s) = \sup_{\{c,k'\}} \quad [U(c) + \beta \cdot E\{V(k',s')|s\}]$

s.t. $\quad c + k' \leq A(s)\, k$

Here, the expression $E\{V(k',s')|s\}$ is the expected value of $V(k',s')$ given that the current value of the state is s. That is, s' is drawn from the conditional distribution of s_{t+1} given s_t which, given that the $\{s_t\}$ is stationary, does not depend on t. Further, note that k' and c are functions of s and k of course, that is the solution to the RHS of Bellman's Equation defines two functions, $c(k,s)$ and $k'(k,s)$, (assuming the the solution to the problem in the right exists and is unique for all (k,s)) and they satisfy:

$$V(k,s) = U(c(k,s)) + \beta \cdot E\{V(k'(k,s),s')|s\}.$$

NOTE: It is useful for students to write out what these equations mean in terms of sums by assuming that the process $\{s_t\}$ is a stationary Markov Chain with transition probabilities $\pi_{s',s} = \Pr(s_{t+1} = s'|s_t = s)$.

NOTE: Strictly speaking the above theorem requires some 'measurability' assumptions too. See SLP for this. These will automatically satisfied if the state space, S, is countable. Since every function defined on a countable set is measurable automatically.

Given the proposition above, we know something about the form of V. This is that it is homogeneous of degree $1-\sigma$ in k. This has implications about the form of Bellman's equation in this example:

$$E(V(k',s')|s) = E((k')^{1-\sigma} V(1,s')|s) = (k')^{1-\sigma} E(V(1,s')|s).$$

Here the last equality comes from the fact that $E(f(X)Y|X) = f(X)E(Y|X)$ for any random variables X and Y and any function f.

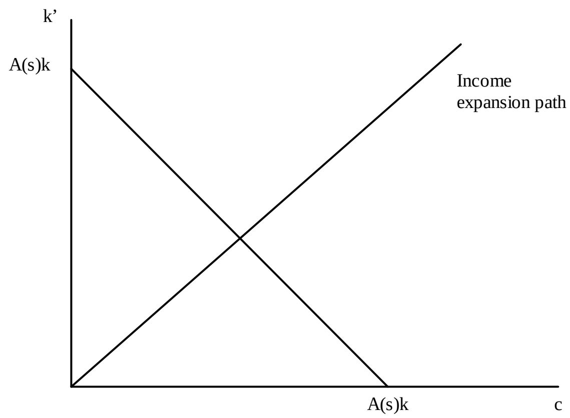
If we make the additional assumption that the s_t process is i.i.d.this simplifies even further because $E(V(1,s')|s) = E(V(1,s'))$. Thus, in this case, we have:

$$E(V(k',s')|s) = (k')^{1-\sigma} E(V(1,s')|s) = (k')^{1-\sigma} E(V(1,s')).$$

Let $D = E(V(1,s))$ and note that this is not random.

From all of this, it follows that in the case under consideration with i.i.d. shocks, that the Right Hand Side of Bellman's Equation simplifies to:

$$\sup_{c,k'} \quad U(c) + \beta(k')^{1-\sigma} D$$



s.t. $\qquad c + k' \leq A(s)k$

Where $D = E(V(1, s))$ is a constant. But, because of our assumption about the form of U, this objective function is homothetic in the choice variables, (c, k'). Hence, as we have seen before, it follows that the solution, which clearly depends only on $A(s)k$ and not on s and k individually, is of the form:

$$(c(k, s), k'(k, s)) = (\varphi A(s)k, (1 - \varphi)A(s)k)$$

for some $0 \leq \varphi \leq 1$. Note in particular that φ, the fraction of output going to consumption, does not depend on the value of the current shock, s or on the size of the current capital stock, k. This is something that is particular to the i.i.d. case. In general, D would depend on s and hence, φ would too.

To finish the solution of the problem and study its properties, all we have to do is to figure out what φ is.

There is more than one way to do this, but probably the simplest is to use the stochastic version of the Euler Equation in conjunction with what we have already learned about the solution from our application of Dynamic Programming above.

The direct first order condition that comes out of $P(k_0, s_0)$ is:

$$U_c(t) = \beta E\{U_c(t+1)r_{t+1} \mid s_t\},$$

where $U_c(t) = \partial U(c(s^t))/\partial c(s^t)$.

In this case, since $U(c) = c^{1-\sigma}/(1-\sigma)$, $U_c(t) = c_t^{-\sigma}$ and hence this can be rewritten as:

$$1 = \beta E\{(c_t/c_{t+1})^{\sigma} r_{t+1} \mid s_t\}.$$

Subsituting that $c_t = \varphi A(s_t)k_t$, we get that

$1 = \beta E\{(c_t/c_{t+1})^{\sigma} r_{t+1} \mid s_t\} = \beta E\{(\varphi A(s_t)k_t/\varphi A(s_{t+1})k_{t+1})^{\sigma} r_{t+1} \mid s_t\} =$

$\beta E\{(A(s_t)k_t/A(s_{t+1})(1-\varphi)A(s_t)k_t)^{\sigma} r_{t+1} \mid s_t\} = \beta E\{(A(s_{t+1})(1-\varphi))^{-\sigma} r_{t+1} \mid s_t\}.$

As is standard in Ak models, $r_{t+1} = A(s_{t+1})$. Using this plus the fact from above that φ is not random if the s_t are i.i.d., we can simplify this to:

$(1-\varphi)^{\sigma} = \beta E\{(A(s_{t+1}))^{-\sigma} A(s_{t+1}) \mid s_t\} = \beta E\{(A(s_{t+1}))^{1-\sigma} \mid s_t\} = \beta E\{(A(s_{t+1}))^{1-\sigma}\}$.

Note that the last step again uses the assumption that the s_t process is i.i.d.

Finally, let's assume for simplicity that $A(s_t) = As_t$. Then, this equation reads:

$(1-\varphi) = [\beta E\{(As)^{1-\sigma}\}]^{1/\sigma}$.

Where we have used the fact that the process is stationary so that $E\{(As_{t+1})^{1-\sigma}\}$ does not depend on t.

Note that this says that the optimal savings rate, $(1-\varphi)$ depends on the entire distribution of s and not just it's mean value. In particular, it is a monotone increasing function of $E\{s^{1-\sigma}\}$

What are the properties of the growth rate of output in this model? To see this, recall that output in period t at node s^t is $A(s_t)k(s^{t-1})$ and hence, we have:

$y(s_{t+1}, k_{t+1})/y(s_t, k_t) = A(s_{t+1})k_{t+1}/A(s_t)k_t = A(s_{t+1})(1-\varphi)A(s_t)k_t/A(s_t)k_t =$

$(1-\varphi)A(s_{t+1})$.

It follows that

$$E(y_{t+1}/y_t) = (1-\varphi)E(A(s_{t+1})) = (1-\varphi)AE(s_{t+1}).$$

Thus, movements in the mean growth rate, are completely determined by the savings rate, $(1-\varphi)$.

The only thing left to determine is the properties of $(1-\varphi)$ as it depends on the distribution of s.

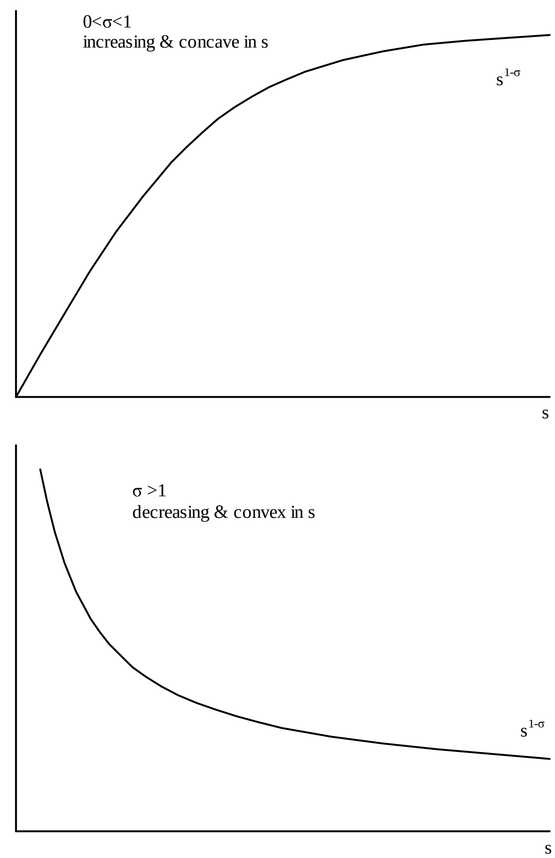
4.3.1 How does $E(\gamma)$ depend on $d\mu(s)$?

As noted, $E(\gamma)$ is a monotone function of φ. (if $\varphi \uparrow \implies E(\gamma) \downarrow$)

- Increasing $E(s^{1-\sigma})$ will decrease φ. This leads to higher savings and $E(\gamma) \uparrow$.

  Decreasing $E(s^{1-\sigma})$ will increase φ. This leads to lower savings and $E(\gamma) \downarrow$.

- Properties of $E(s^{1-\sigma})$ as a function of $d\mu(s)$.


0<σ<1
increasing & concave in s
s1-σ
s
σ >1
decreasing & convex in s
s1-σ
s

<Jensen's Inequality- A Technical Aside> i) If g is convex, $g(E(s)) \leq E(g(s))$,

ii) If g is concave, $g(E(s)) \geq E(g(s))$.

Note: Equality holds if g is linear in s or if the distribution of s puts mass one on one point.

From above, $s^{1-\sigma}$ is increasing and concave in s if $0 < \sigma < 1$, and it is decreasing and convex in s if $1 < \sigma$.

Without loss of generality, let's normalize so that $E(s) = 1$ (the rest can be put into A.)

So, if $\sigma > 1$, $E(s^{1-\sigma}) \geq (E(s))^{1-\sigma} = 1$.

Note also that $E(s)^{1-\sigma} = E(s^{1-\sigma})$ if $\mu(ds) = 1$ no matter what the value of σ.

It follows that $E(s^{1-\sigma})$ is larger (weakly) under any other distribution for s with $E(s) = 1$ than it is under the distribution $\mu(ds) \equiv 1$.

That is, consider two possible distributions for s.

$\mu_1(ds) \equiv$ any distribution of s's with $E(s) = 1$

$\mu_2(ds) \equiv$ Larry's choice, $s = 1$ with probability one. i.e. $P(s = 1) = 1$

Let $\gamma_1 \equiv$ expected growth rate if the s's are i.i.d. $\mu_1(ds)$ and $\gamma_2 \equiv$

expected growth rate if s's are i.i.d. $\mu_2(ds)$

Then, we want to know which is larger γ_1 or γ_2?

From above, it follows that

$$\gamma_1 > \gamma_2 \iff \int s^{1-\sigma} d\mu_1 > 1$$

$$\gamma_1 < \gamma_2 \iff \int s^{1-\sigma} d\mu_1 < 1.$$

From Jensen's Inequality,

$$Eg(s) = \int s^{1-\sigma} d\mu_1(ds) > 1 = g(E(s)) \quad iff \quad \sigma > 1, \text{ and}$$

$$Eg(s) = \int s^{1-\sigma} d\mu_1(ds) < 1 = g(E(s)) \quad iff \quad \sigma < 1.$$

Thus,

Proposition:

i) $\gamma_1 > \gamma_2 \iff \sigma > 1,$

ii) $\gamma_1 < \gamma_2 \iff \sigma < 1.$

That is, the growth rate is higher under uncertainty than under certainty if and only if $\sigma > 1$, and it is lower under uncertainty than under certainty if and only if $\sigma < 1$.

Summarizing: Under high risk aversion, adding uncertainty increases the savings rate and hence the growth rate, while the opposite occurs if risk aversion is low.

4.3.2 Remarks:

1. If $\sigma = 1$, the case of log utility, the growth rate is the same under certainty and uncertainty.

2. Under $U(c) = c$, risk neutrality, $\sigma = 0$, adding uncertainty lowers the growth rate.

3. Under Leontieff Preferences, $\sigma = \infty$, adding uncertainy increases the growth rate.

4. Intuitively, if $1-\varphi$ increases, you are saving more for the future. In this way, you are acting in the only way you can to provide self-insurance. This, increases the growth rate.

Can this result be generalized beyond the comparison between perfect certainty and uncertainty? Yes.

Definition: Z_2 is a **mean preserving spread** over Z_1 if

1) $Z_2 = Z_1 + \varepsilon$

2) $E(\varepsilon) = 0$

3) ε and Z_1 are independent.

Intuitively, Z_2 is a **mean preserving spread** over Z_1 if it is 'noisier' than Z_1.

Example: If Z_1 is such that, $P(Z_1 = 1) = 1$ (like γ_2 case), then for any Z_2 s.t. $E(Z_2) = 1$ (like γ_1 case), is a mean preserving spread over Z_1.

Theorem: (Rothschild and Stiglitz) If Z_2 is a **mean preserving spread** over Z_1 then

i) $E(g(Z_1)) > E(g(Z_2)) \quad \forall$ concave g, and,

ii) $E(g(Z_1)) < E(g(Z_2)) \quad \forall$ convex g.

In fact, a converse of this also holds. See the Rothschild and Stiglitz paper for details.

Since we already have the characterization of $E(\gamma)$ in our case in terms of $E(s^{1-\sigma})$, and we know which way $E(s^{1-\sigma})$ goes as a function of σ, we have the following result:

Theorem: If $\mu_1(ds)$ & $\mu_2(ds)$ are the distributions of the shocks in two $\tilde{A}k$ models and μ_2 is a mean preserving spread over μ_1,

then, $E(\gamma_2) > E(\gamma_1) \quad$ if $\sigma > 1$

$E(\gamma_2) < E(\gamma_1) \quad$ if $0 < \sigma < 1$

Proof: Obvious.

An interesting special case of this result when we remember that if the Budget is Balanced period by period (and here, state by state as well), models with linear income taxes are equivalent to those with altered production functions:

- Consider the model where A is certain but there is a random income tax and random government spending with the government balancing its budget in a state by state way:

Consumer's Problem:

$$Max \qquad E_0 \left[\sum \beta^t U\left(c\left(s^t\right)\right) \right]$$

s.t. $\sum \sum p_t\left(s^t\right)\left[c_t\left(s^t\right) + x_t\left(s^t\right)\right] \leq \sum \sum \left[\left(1 - \tau\left(s^t\right)\right) r_t\left(s^t\right) k_t\left(s^t\right)\right]$

$k_{t+1}\left(s^t\right) \leq x_t\left(s^t\right)$

where full depreciation is assumed as above and k_0 is given.

Firms Problem:

$$Max \qquad p_t\left(s^t\right)\left[c_t^f\left(s^t\right) + x_t^f\left(s^t\right) + g_t^f\left(s^t\right)\right] - r_t\left(s^t\right) k_t^f\left(s^t\right)$$

s.t. $\qquad c_t^f\left(s^t\right) + x_t^f\left(s^t\right) + g_t^f\left(s^t\right) \leq A k_t^f\left(s^t\right)$.

Finally, we assume that $p_t\left(s^t\right) g_t\left(s^t\right) = \tau_t\left(s^t\right) A k_t^f\left(s^t\right)$ for all t and s^t.

(Note that $k_t^f(s^t) = k_t(s^{t-1})$ in equilibrium, so that although it looks like the firms capital stock is a function of the current state, in equilibrium it cannot be since it is equal to that of the consumer and this is determined as a function of actions taken in period $t-1$.)

The equilibrium of this model solves

$$\begin{aligned} &Max \qquad E_0\left[\sum \beta^t U(c_t(s^t))\right] \\ &\text{s.t.} \qquad c_t(s^t) + x_t(s^t) \leq (1-\tau_t(s^t))\,Ak_t(s^{t-1}) \\ &\qquad\qquad k_{t+1}(s^t) \leq x_t(s^t) \end{aligned}$$

- Consider two alternative fiscal policy, $\tau_1(s^t)$ and $\tau_2(s^t)$ such that:

  i) $E(\tau_1(s^t)) = E(\tau_2(s^t))$

  ii) $\tau(s^t)'$s are i.i.d.

  iii) $U = \frac{c^{1-\sigma}}{1-\sigma}$

  Then, when $\sigma > 1,\ E(\gamma \text{ under } \tau_1) > E(\gamma \text{ under } \tau_2)$ if τ_1 is a mean preserving spread over τ_2.

Note: If you want higher growth then make income taxes more volatile!?!?!?!?!?! An interesting counter-intuitive result that serves as a good place to stop.